Assignment 2

Hand in no. 3, 4 and 8 by September 20, 2018.

1. This is an optional problem.
   (a) Assume that the Fourier coefficients of a continuous, 2π-periodic function vanish identically. Show that this function must be the zero function. Hint: WLOG assume $f(0) > 0$. Use the relation
   $$\int_{-\pi}^{\pi} f(x)p(x)dx = 0,$$
   where $p(x)$ is a trigonometric polynomial of the form $(\varepsilon + \cos x)^k$ for some small ε and large $k > 0$.
   (b) Use the result in (a) to show that if the Fourier series of a continuous, 2π-periodic function converges uniformly, then it converges uniformly to the function itself.
   (c) Apply (b) to the Fourier expansion of x^2 to show that
   $$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$
2. Let f be a complex valued 2π-periodic function whose derivative is again integrable on $[-\pi, \pi]$. Show that c_n and c_n', the Fourier coefficients of f and f' respectively, satisfies the relation $c_n' = inc_n, n \in \mathbb{Z}$. Do not do it formally. Use the definition of the integration of complex valued functions.
3. Let $C_{2\pi}^{\infty}$ be the class of all smooth 2π-periodic, complex-valued functions and $\mathcal{C}^{\infty}$ the class of all complex bisequences satisfying $c_n = \circ(n^{-k})$ as $n \to \pm\infty$ for every k. Show that the Fourier transform $f \mapsto \hat{f}$ is bijective from $C_{2\pi}^{\infty}$ to $\mathcal{C}^{\infty}$.
4. Propose a definition for $\sqrt{d/dx}$. This operator should be a linear map which maps smooth functions to smooth functions satisfying
   $$\sqrt{\frac{d}{dx}}\sqrt{\frac{d}{dx}}f = \frac{d}{dx}f,$$
   for all smooth, 2π-periodic f.
5. Let f be a continuous, 2π-periodic function and its primitive function be given by
   $$F(x) = \int_0^x f(x)dx.$$
   Show that F is 2π-periodic if and only if f has zero mean. In this case,
   $$\hat{F}(n) = \frac{1}{in}\hat{f}(n), \quad \forall n \neq 0.$$
6. Let $\mathcal{C}'$ be the subspace of $\mathcal{C}$ consisting of all bisequences $\{c_n\}$ satisfying $\sum_{-\infty}^{\infty} |c_n|^2 < \infty$.
   (a) For $f \in R[-\pi, \pi]$, show that
   $$\sum_{-\infty}^{\infty} |c_n|^2 \leq \int_{-\pi}^{\pi} |f|^2 .$$

(b) Deduce from (a) that the Fourier transform $f \mapsto \hat{f}(n)$ maps $R_{2\pi}$ into $\mathcal{C}'$.

(c) Explain why the trigonometric series

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^\alpha}\ , \quad \alpha \in (0, 1/2]\ ,$$

is not the Fourier series of any function in $R_{2\pi}$.

7. Let f be a C^1-piecewise, continuous 2π-periodic function. In other words, there exist $-\pi = a_1 < a_2 < \cdots < a_N = \pi$ and C^1-functions f_j defined on $[a_j, a_{j+1}],\ j = 0, \cdots, N-1$ such that $f = f_j$ on (a_j, a_{j+1}). Show that its Fourier series converges uniformly to itself. Hint: Let $M = \max_j\{\sup |f_j'(x)| :\ x \in [a_j, a_{j+1}]\}$. Establish $|f(y) - f(x)| \le M|y - x|$ for all $x, y \in [-\pi, \pi]$.

8. Show that for a Lipschitiz continuous, 2π-periodic function, its Fourier coefficients satisfy

$$|a_n| \le \frac{C\pi}{n}, \quad |b_n| \le \frac{C\pi}{n}\ ,$$

for some constant C.